Superpotentials of $N = 1$ Supersymmetric Gauge Theories from M-theory

Soonkeon Nam[§], Kyungho Oh[∗], Sang-Jin Sin[†]

[§] *Dept. of Physics, Kyung Hee University, Seoul 130-701, Korea*

nam@nms.kyunghee.ac.kr

[∗] *Dept. of Mathematics, University of Missouri-St. Louis,*
St. Louis, Missouri 63121, USA

oh@arch.umsl.edu

[†] *Dept. of Physics, Hanyang University, Seoul 133-791, Korea*

sjs@dirac.hanyang.ac.kr

Abstract

We consider brane configurations in M-theory describing $N = 1$ supersymmetric gauge theories and using the parametric representation of the brane configurations, we calculate the superpotentials for various cases including multiple gauge groups or fermions. For $SU(n)$ $N = 1$ SQCD with N_f fermion case ($N_f < N_c$), we find that the superpotential from M-theory and the gauge theory agree precisely. This gives a direct evidence of the validity of Witten's M-theory method for calculating the superpotential.

1 Introduction

The idea of D-brane[1] opened up a new and surprisingly simple way to communicate between super symmetric (SUSY) gauge theories and the superstring theories. Many intriguing results about the field theories have been achieved by investigating gauge theories formulated on the world-volume of branes in string theories[2-11] and M-theory[12-16]. In a recent paper [12], Witten provided solutions of $N = 2$ SUSY gauge theories in four dimension[17, 18] by reinterpreting configurations of fourbranes and fivebranes in type II superstring as the branes in M theory. Moreover, in a subsequent paper[15], he showed how some of the outstanding problems in particle physics such as quark confinement and chiral symmetry breaking, can be approached from the M-theory point of view. There, he also suggested a way to calculate the superpotential directly from a brane configuration, thereby he gave a direct evaluation of the tension of the domain wall[19]. Though very insightful, no direct comparison with the gauge theory result was given.

One of the main goal of this paper is to give a direct evidence for the validity of Witten's method by extending his result to the cases where gauge theory results are available. We will show that the minimum of the superpotential of the $N = 1$ $SU(n)$ gauge theory superpotential agrees with the value evaluated with the M-theory method. We will also generalize and obtain the superpotentials for more general situations with product gauge groups, which corresponds to the brane configurations where there are $n + 1$ fivebranes. To motivate the study of multiple branes we recall the works of Elitzur et al. who have considered k 'coincident' NS fivebranes linked to an 'coincident' NS fivebrane linked by N_c D fourbranes[5].

2 The Brane Configuration

To fix the notation, we begin with brane configurations in type IIA superstring theory. Let the spacetime coordinates be $(x^0, x^1, \ldots, x^9)$ on [10]. Our basic brane configuration contains three kinds of branes: NS fivebranes whose world volume occupies $(x^0, x^1, x^2, x^3, x^4, x^5)$ spacetime coordinates, D fourbranes occupying $(x^0, \cdots, x^3, x^6)$,

and D sixbranes occupying $(x^0, \cdots, x^3, x^7, x^8, x^9)$.

The basic brane configuration consists of N_c fourbranes suspended between two parallel fivebranes in Type IIA superstring theory which gives a representation of $N = 2$ SUSY QCD (SQCD) in four dimensions $(x^0, \cdots, x^3)$ with gauge group $SU(N_c)$. Also N_f D sixbranes could be added and this gives N_f hypermultiplets in the fundamental representation of the gauge group $SU(N_c)$.

One of the limitation of type IIA superstring theory is that a fourbrane ending on a fivebrane defies any explicit conformal field theory description. However, by going to M theory, the picture becomes more unified. A type IIA fourbrane becomes an M theory fivebrane that is wrapped over the 11th dimension [1]. Thus D fourbranes and NS fivebranes come from the same basic object in M theory. This $N = 2$ SUSY description can be broken to $N = 1$ by turning on a mass for the adjoint chiral superfield in the $N = 2$ vector multiplet.

Let us now review some of the Witten's construction[12, 15] of the parametric representation of the brane configuration in M theory for $N = 1$ SQCD. To set up the brane configurations in complex geometry, we introduce complex coordinates $v = x^4 + ix^5$, $w = x^7 + ix^8$, $t = \exp(-(x^6 + ix^{10})/R)$. The brane configuration for $SU(n)$ gauge theory is given by the Seiberg-Witten curve[17]

$$t^2 + P_n(v)t + 1 = 0, \tag{1}$$

in $v - t$ space when $w = 0$. Here $P_n(v)$ is a polynomial of the form

$$P_n(v) = v^n + u_2 v^{n-2} + \ldots + u_n, \tag{2}$$

where the u_i's are the "order parameters" of the theory.

The brane configuration associated to $N = 1$ SUSY can be obtained by "rotating" the brane configuration that describes $SU(n)$ gauge theory with $N = 2$ SUSY, in the sense of Ref.[21]. The rotation is possible only when all the 1-cycles on the corresponding curve are degenerate. This is why the $N = 1$ curve is rational for the two NS fivebrane case[15]. However for more than two NS fivebranes, we can ask whether the curve is generically rational. By reversing the argument we can say

that $N = 2$ SUSY forbids the rotation of the NS fivebranes. Once $N = 2$ SUSY is broken, however, there is nothing that freezes the rotating degrees of freedom. This means that in generic configuration the NS fivebranes are rotated to one another by certain angles. Therefore the curves with positive genus correspond very special non-generic cases. Hence the generic M-brane configuration corresponds to a curve Σ of genus zero, which has parametric representations. Namely, Σ can be identified by a holomorphic mapping with a Riemann sphere with some points deleted. If λ is the complex coordinate of Σ, the holomorphic functions $v(\lambda), w(\lambda)$, and $t(\lambda)$ on Σ are given by rational functions of λ.

Since v has only poles at ends of fivebranes, we can put $v = \lambda + c\lambda^{-1}$, for some constant c. Also, t can go to zero or infinity only at poles of v, so t is a constant multiple of a power of λ, i.e. $t \sim \lambda^n$. We also get $c^n = 1$ from eq.(1), and for each choice of c the polynomial P_n is uniquely determined. After the rotation, w, instead of being zero, should be a non-zero holomorphic function on Σ. Rotating only one fivebrane, w should get a pole only at one end of Σ and should vanish at the other end. So we can set $w = \zeta\lambda^{-1}$ for some complex constant ζ. As λ goes to zero $w/v \to \zeta/c$, therefore it can be regarded as a tangent of the rotation angle, namely

$$\tan\theta = |\zeta/c|. \tag{3}$$

For the rotation of $\theta = \pi/2$, one should set $c = 0$, because v and w do not have a common pole. Therefore the parametric equation for the curve becomes; $v = \lambda$, $w = \zeta\lambda^{-1}$, $t = \lambda^n$.

Now we give a parametric representation of the brane configuration with more than two NS fivebranes. Suppose NS fivebranes are located at $\lambda = \lambda_\alpha$, $\alpha = 0, 1, \cdots, n$ at arbitrary angles in $v-w$ space, with k_α D fourbranes suspended between $\alpha - 1^{\text{th}}$ and α^{th} fivebranes. Thus the functions v and w on Σ have poles at λ_α, $\alpha = 0, 1, \cdots, n$. The function t on Σ will have a zero or pole according to the direction the NS fivebrane is bent. If there are more D fourbranes in the left than right it will bend to the right, therefore it will have a zero and vice versa. In fact $t \sim v^{a_\alpha}$ as $v \to \infty$ where

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$a_\alpha = k_{\alpha+1} - k_\alpha$. Putting all these together, we have

$$v = \sum_{\alpha=0}^{n} \frac{c_\alpha}{\lambda - \lambda_\alpha},$$

$$w = \sum_{\alpha=0}^{n} \frac{s_\alpha}{\lambda - \lambda_\alpha},$$

$$t = \prod_{\alpha=0}^{n} (\lambda - \lambda_\alpha)^{-a_\alpha}, \tag{4}$$

for some complex numbers c_α, s_α. We normalize the system by setting $c_0 = 1$, $s_0 = 0$, so that λ_0 is the location of the unrotated NS fivebrane. Then the rotation angles θ_α's are again given by

$$\tan \theta_\alpha = \left| \frac{s_\alpha}{c_\alpha} \right|. \tag{5}$$

Notice that Witten's choice corresponds to the special case where two poles are chosen as $\lambda = 0$ and ∞ and with rotation angle $\pi/2$. Since it can be related to the $n = 1$ case of above with $(\lambda_0, \lambda_1) = (0, 1)$ by the Möbius transformation $\lambda \to \tilde{\lambda} = \frac{\lambda - 1}{\lambda}$ followed by constant shifts in v and w, they are physically equivalent.

When one approaches the ends of the fivebranes, v and w go to infinity. Since the effect of the rotation can be ignored, the equation for the $N = 2$ Seiberg-Witten curve

$$t^{n+1} + P_{k_1}(v)t^n + \cdots + 1 = 0,$$

must be satisfied near the ends. This determines some of the moduli of the $N = 1$ curve in terms of the moduli of the polynomial equation for $N = 2$ case. As λ approaches λ_α the parametrization of the curve is asymptotically given by,

$$v \to \frac{c_\alpha}{(\lambda - \lambda_\alpha)},$$

$$w \to \frac{s_\alpha}{(\lambda - \lambda_\alpha)},$$

$$t \to (\lambda - \lambda_\alpha)^{-a_\alpha} \prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)^{-a_\beta}. \tag{6}$$

Inserting these to eq.(6), we get the following:

$$c_\alpha^{a_\alpha} = -\frac{p_{\alpha+1}}{p_\alpha} \prod_{\beta \neq \alpha} (\lambda_\alpha - \lambda_\beta)^{-a_\beta}, \quad (\alpha = 0, \cdots, n),$$

where p_α denotes the coefficient of the leading power in $P_{k_\alpha}(v) = p_\alpha v^{k_\alpha} + \cdots$. This determines the coefficients of the poles of v as function of the location of the fivebranes and the information of $N = 2$ theory up to the phase $e^{2\pi i/(k_{\alpha+1}-k_\alpha)}$. As Witten argued, p_α's are parameters of the Lagrangian rather than a modulus, so we conclude that only the locations of the NS fivebranes λ_α's and the parameter for angles of the $\alpha+1^{\text{th}}$ NS brane relative to the first one, s_α's, are the moduli of the $N = 1$ theory.

Let us now discuss the symmetry of the curve. Although there is no remaining symmetry for interacting branes, asymptotically, when λ is close to λ_α, we have following symmetry:

$$\prod_{\alpha=1}^{n} \mathbf{Z}_{|a_\alpha|} = \mathbf{Z}_{k_2-k_1} \times \cdots \times \mathbf{Z}_{k_{n-1}-k_n} \times \mathbf{Z}_{k_n}.$$

This corresponds to the rotational symmetries of the branes when v goes to infinity broken to the discrete symmetries due to the quantum effects. We remark that this is the analogue of chiral symmetry for $SU(n)$ case in which the chiral symmetry is $\mathbf{Z}_n$. The structure of the symmetry manifestly reveal the consequence of the interactions between the branes. The field theory analogues of this effect for the product gauge groups are not fully studied. One technical remark is that in considering the symmetry properties, it is convenient to locate the zeroth brane at the infinity so that it looks special relative to the others.

3   The Superpotential

Given the brane configurations in M-theory, we calculate the superpotential of the corresponding gauge theory following Witten's idea[15]. Consider in general M-theory compactification on $\mathbf{R}^4 \times X \times \mathbf{R}$ where X is a Calabi-Yau threefold. Suppose in spacetime there are fivebranes of the form $\mathbf{R}^4 \times \Sigma$, Σ being a two-dimensional real surface in X. Choose Σ_0 in the homology class of Σ in $H^2(X, \mathbf{R})$. Then there exists a three manifold B and a map $\Phi_B : B \to X$ such that the boundary of B maps $\Sigma - \Sigma_0$ in X. Let Ω be the holomorphic three form on X. Then as Witten suggested, the

superpotential is given by

$$W(\Sigma) - W(\Sigma_0) = \int_B \Phi_B^*(\Omega). \tag{7}$$

This defines $W(\Sigma)$ up to an additive constant. The ambiguity in $W(\Sigma)$ comes from the choices of B and Σ_0. In general, the periods of Ω also contribute to the ambiguity, but this indeterminancy disappears since $H_3(X,) = 0$ in our applications. The condition $H_3(X,) = 0$ also forces that the space of all possible Σ is simply-connected. Thus a different choice of B does not create any additional constant.

To discuss the issue of the Σ_0 dependence, we consider two NS fivebranes connected by n D fourbranes and assume that two are at a relative angle θ. We take X to be a flat Calabi-Yau manifold Y with coordinates v, w, and t. The holomorphic three form Ω on Y is given by

$$\Omega = R\frac{dt}{t} \wedge dv \wedge dw. \tag{8}$$

It is chosen such that $\Omega \wedge \overline{\Omega}$ is the Riemannian volume form for Y. Since we can choose the position of a fivebrane at $\lambda = 0$ without loss of generality, the configuration is parametrized by

$$\begin{aligned}
v(\lambda) &= \frac{1}{\lambda - \lambda_1} + \frac{c}{\lambda}, \\
w(\lambda) &= \frac{\zeta}{\lambda}, \\
t(\lambda) &= \left(\frac{\lambda}{\lambda - \lambda_1}\right)^n.
\end{aligned} \tag{9}$$

In fact the dependence of the superpotential on the choice of the Σ_0 can be a subtle issue. In principle, fixing a Σ_0 just amounts to fixing the zero point of the superpotential. In practice, however, for a given Σ one has to choose an appropriate Σ_0 to expedite calculation. Therefore we want to put $W(\Sigma_0) = 0$ for a certain class of surfaces. In the next paragraph, for the fixed Σ given by eq.(12), we choose $\tilde{\Sigma}_0$ and Σ_0 which look completely different, and show by explicit calculation that they in fact give the same superpotential.

To construct $\tilde{\Sigma}_0$ we first introduce a new variable $\tilde{\lambda} = \frac{\lambda}{\lambda - \lambda_1}$ and take Σ_λ to be the complex λ-plane with 0 and λ_1 deleted. We write $\tilde{\lambda} = \exp(\tilde{\rho} + i\tilde{\theta})$, with $\tilde{\rho}$ and

$\tilde{\theta}$ real, and pick an arbitrary smooth function $\tilde{f}$ of a real variable such that $\tilde{f}(\tilde{\rho}) = 1$ for $\tilde{\rho} > 2$ and $\tilde{f}(\tilde{\rho}) = 0$ for $\tilde{\rho} < 1$. Then we define the map $\Phi_0 : \Sigma_\lambda \to Y$ by

$$v = \frac{1}{\lambda - \lambda_1} \tilde{f}(\tilde{\rho}) + \frac{c}{\lambda} \tilde{f}(-\tilde{\rho}),$$

$$w = \tilde{f}(-\tilde{\rho}) \frac{\zeta}{\lambda},$$

$$t = \left(\frac{\lambda}{\lambda - \lambda_1} \right)^n. \tag{10}$$

By construction $\tilde{\Sigma}_0$ is asymptotic at infinity to Σ. We now introduce smooth bounded functions $g_\pm = g_\pm(\tilde{\rho}, \sigma)$ given by $g_+(\tilde{\rho}, 1) = 1$, $g_+(\tilde{\rho}, 0) = f(\tilde{\rho})$, and $g_+(\tilde{\rho}, \sigma) = 1$, and $g_-(\tilde{\rho}, \sigma) = g_+(-\tilde{\rho}, \sigma)$. The map $\Phi_B : B \to Y$ can then be defined by

$$v = \frac{1}{\lambda - \lambda_1} g_+(\tilde{\rho}, \sigma) + \frac{c}{\lambda} g_-(\tilde{\rho}, \sigma),$$

$$w = \frac{\zeta}{\lambda} g_-(\tilde{\rho}, \sigma),$$

$$t = \tilde{\lambda}^n. \tag{11}$$

The superpotential now becomes

$$W(\Sigma) - W(\tilde{\Sigma}_0) = Rn \int_B \frac{d\tilde{\lambda}}{\tilde{\lambda}} \wedge dv \wedge dw, \tag{12}$$

and thus

$$W(\Sigma) - W(\tilde{\Sigma}_0) = iRn \int_0^1 d\sigma \int_0^{2\pi} d\tilde{\theta} \int_{-\infty}^{\infty} d\tilde{\rho} \frac{\zeta}{\lambda_1^2} \frac{(\tilde{\lambda} - 1)^2}{\tilde{\lambda}} \left(\frac{\partial g_+}{\partial \sigma} \frac{\partial g_-}{\partial \rho} - \frac{\partial g_+}{\partial \rho} \frac{\partial g_-}{\partial \sigma} \right). \tag{13}$$

Notice that the $g_\pm$ is independent of $\tilde{\theta}$. Therefore the integrand is sum of the terms whose integrals split into $\int d\tilde{\theta}$ and the rests. The final result is

$$W(\Sigma) = -\frac{4\pi i Rn\zeta}{\lambda_1^2}. \tag{14}$$

Now we choose a different Σ_0 given by the Φ_0 which is determined by continuous and smooth functions f_0, f_1. Let $\lambda/\lambda_1 = \exp(\rho + i\theta)$ and construct f_i's such that $f_0(\rho) = 1$ if $\rho < -3$, $f_0 = 0$ if $\rho > -2$ and $f_1(\rho) = 1$ if $|\rho - 1| < 1$, $f_1 = 0$ if $|\rho - 1| > 2$. Construct the map $\Phi_0 : \Sigma_\lambda \to Y$ just as above by replacing $\tilde{f}(\rho)$ and $\tilde{f}(-\rho)$ by $f_i(\rho)$'s. Then construct g_i, $i = 0, 1$ such that they interpolate f_i and

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constant function 1 continuously and smoothly. Then we choose the map $\Phi_B : B \to Y$ as above by replacing $g_\pm$ by g_i's. Then the superpotential now becomes

$$W(\Sigma) - W(\Sigma_0) = iRn \int \left[\frac{1}{\lambda^2(\lambda - \lambda_1)} - \frac{1}{\lambda(\lambda - \lambda_1)^2} \right] d\lambda \wedge dg_1 \wedge dg_0. \qquad (15)$$

This integral also can be evaluated easily by noticing that g_i's do not depend on the angular variable θ, and is exactly the same as the previous case. Although the bases of f's and $\tilde{f}$'s, defined as the regions in Σ_λ where Σ_0 and $\tilde{\Sigma}_0$ respectively is equal to Σ, look very different, Σ_0 and $\tilde{\Sigma}_0$ has a common property. They both asymptotically approach to Σ and have necks that have zero thickness which makes the n invariance more manifest. The general construction described below will respect this feature.

Notice that the superpotential is independent of the rotation angle for the two fivebrane cases. There are overall factor 2 difference between the previous result with that for Witten's configuration, which gives 2π rather than 4π. This can be understood by noticing that in Witten's choice, the pole at the infinity is truncated from the expression of t so that the contribution from that pole is not included.

Now let us calculate the superpotential for the general multi-brane cases, given by eq.(4). We assume that $|\lambda_0| > |\lambda_1| > \cdots > |\lambda_n|$. Let $\rho = \log|\lambda|$, $\rho_\alpha = \log|\lambda_\alpha|$ for $\alpha = 0, \cdots, n$, and $\epsilon = \min_{\alpha=0,\ldots,n}(\rho_\alpha - \rho_{\alpha+1})$. We construct Σ_0 in terms of the functions $f_\alpha(\rho)$ which are defined by

$$\begin{aligned} f_\alpha(\rho) &= 1 \quad \text{if} \quad |\rho - \rho_\alpha| < \frac{1}{3}\epsilon, \\ &= 0 \quad \text{if} \quad |\rho - \rho_\alpha| > \frac{2}{3}\epsilon, \end{aligned} \qquad (16)$$

and f_α interpolate 0 and 1 in the region $\frac{1}{3}\epsilon < |\rho - \rho_\alpha| < \frac{2}{3}\epsilon$. We define the homotopy functions $g_\alpha(\rho, \sigma)$'s which interpolate $f_\alpha(\rho)$ and 1 such that $g_\alpha(\rho, 0) = f_\alpha(\rho)$, $g_\alpha(\rho, 1) = 1$. In other words, each f_α has a circular strip containing the circle passing the λ_α as its territory where it is 1 and rapidly dies outside and no territories are overlapping so that $f_\alpha f_\beta = 0$ for any pair. Furthermore for later purpose we construct the g_α's such that

$$g_\alpha(\rho_\gamma, \sigma) = g_\beta(\rho_\gamma, \sigma) := h_\gamma(\sigma), \qquad (17)$$

for any triple α, β, γ which are different to one another. Now take a three manifold B as the product of the λ plane with all the α's deleted and the σ interval $0 \leq \sigma \leq 1$. Then define a map $\Phi : B \to Y$ by

$$v = \sum_{\alpha=0}^{n} \frac{c_\alpha}{\lambda - \lambda_\alpha} g_\alpha(\rho, \sigma),$$

$$w = \sum_{\alpha=0}^{n} \frac{s_\alpha}{\lambda - \lambda_\alpha} g_\alpha(\rho, \sigma),$$

$$t = \prod_{\alpha=0}^{n} (\lambda - \lambda_\alpha)^{-a_\alpha}. \tag{18}$$

Now the superpotential can be written down as

$$W = W(\Sigma) - W(\Sigma_0) = R \int \frac{dt}{t} \wedge dv \wedge dw$$

$$= R \sum_{\alpha,\beta,\gamma} c_\alpha s_\beta a_\gamma \int dg_\alpha \wedge dg_\beta \wedge d\lambda I_{\alpha,\beta,\gamma}(\lambda), \tag{19}$$

where

$$I_{\alpha,\beta,\gamma}(\lambda) = \frac{1}{(\lambda - \lambda_\alpha)(\lambda - \lambda_\beta)(\lambda - \lambda_\gamma)}.$$

Let $\lambda = |\lambda|e^{i\theta}$ and $z = e^{i\theta}$. Then we can first evaluate the $d\lambda$ integral as $d\theta$ integral, which in turn can be evaluated as contour integral on z plane along the unit circle. Therefore

$$\int d\lambda I_{\alpha,\beta,\gamma}(\lambda) = \int |\lambda| dz \frac{1}{(|\lambda|z - \lambda_\alpha)(|\lambda|z - \lambda_\beta)(|\lambda|z - \lambda_\gamma)}. \tag{20}$$

Due to the presence of the homotopy factors, one can avoid $\lambda_\alpha = \lambda_\beta$ However, we can have $\lambda_\gamma = \lambda_\alpha$ or $\lambda_\gamma = \lambda_\beta$.

- The case with $\alpha \neq \gamma \neq \beta$

$$\int d\lambda I_{\alpha,\beta,\gamma}(\lambda) = 2\pi i [\Lambda_\alpha \Theta(\rho - \rho_\alpha) + \Lambda_\beta \Theta(\rho - \rho_\beta) + \Lambda_\gamma \Theta(\rho - \rho_\gamma)], \tag{21}$$

where Λ_i is the residue of $I_{\alpha,\beta,\gamma}(\lambda)$ at λ_i with $i = \alpha, \beta, \gamma$, and Θ denotes the usual step function.

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- The cases $\gamma = \alpha$ or $\gamma = \beta$

$$\int d\lambda I_{\alpha,\beta,\alpha}(\lambda) = \int |\lambda| dz \frac{1}{(|\lambda|z - \lambda_\alpha)^2(|\lambda|z - \lambda_\beta)}$$

$$= \frac{1}{\lambda_{\alpha\beta}^2}[-\Theta(\rho - \rho_\alpha) + \Theta(\rho - \rho_\beta)], \qquad (22)$$

where $\lambda_{\alpha\beta} = \lambda_\alpha - \lambda_\beta$.

Remembering the ordering of ρ_α we can now write the superpotential as

$$W = -2\pi i R \sum_{\alpha<\beta}(c_\alpha s_\beta - c_\beta s_\alpha)\int dg_\alpha \wedge dg_\beta \left[\frac{(a_\alpha - a_\beta)}{\lambda_{\alpha\beta}^2}\Theta_{\alpha\beta}(\rho)\right.$$

$$\left. + \sum_{\gamma\neq\alpha,\beta}^{n} a_\gamma\left(\Lambda_\alpha\Theta(\rho - \rho_\alpha) + \Lambda_\beta\Theta(\rho - \rho_\beta) + \Lambda_\gamma\Theta(\rho - \rho_\gamma)\right)\right], \qquad (23)$$

where $\Theta_{\alpha\beta}(\rho)$ is the step function that is 1 between ρ_α and ρ_β and 0 otherwise. We can also evaluate the rest of the integral. From the construction of the homotopy functions we notice following fact,

$$\int_{C_\gamma} g_\alpha dg_\beta = \delta_{\alpha\gamma},$$

where C_γ is a line along the $\rho = \rho_\gamma$ as σ varies from 0 to 1. By this and the Stokes' theorem it is easy to evaluate the integral

$$\int_{\gamma\delta} dg_\alpha \wedge dg_\beta = \frac{1}{2}(\delta_{\alpha\gamma} + \delta_{\beta\delta} - \delta_{\alpha\delta} - \delta_{\beta\gamma}),$$

where the integral is over a band of the strip defined by $\rho_\delta < \rho < \rho_\gamma, 0 \leq \sigma \leq 1$. Thus, the final result is

$$W = -2\pi i R \sum_{\alpha<\beta}(c_\alpha s_\beta - c_\beta s_\alpha)\left[\frac{(a_\alpha - a_\beta)}{\lambda_{\alpha\beta}^2} + \frac{1}{2}\frac{1}{\lambda_{\alpha\beta}}\sum_{\gamma\neq\alpha,\beta} a_\gamma\left(\frac{1}{\lambda_{\gamma\alpha}} + \frac{1}{\lambda_{\gamma\beta}}\right)\right],$$

where α, β, γ run from 0 to n.

If we choose the parametrization where first NS fivebrane is identified with the large λ region, it is given by

$$v = \lambda + \sum_{\alpha=1}^{n}\frac{c_\alpha}{\lambda - \lambda_\alpha},$$

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$$w = \sum_{\alpha=1}^{n} \frac{s_\alpha}{\lambda - \lambda_\alpha},$$

$$t = \prod_{\alpha=1}^{n} (\lambda - \lambda_\alpha)^{-a_\alpha}. \tag{24}$$

Notice that the $\alpha = 0^{\text{th}}$ components in w and t are truncated. Sometimes, this parametrization makes the discussion of the physics more intuitive. For example, when we discuss the rotations of the branes, one of the branes must be fixed and this distinguished one is located at $\lambda = \infty$. Almost the same calculation gives us the superpotential,

$$W = -2\pi i R \sum_{\alpha < \beta} (c_\alpha s_\beta - c_\beta s_\alpha) \left[\frac{(a_\alpha - a_\beta)}{\lambda_{\alpha\beta}^2} + \frac{1}{2} \frac{1}{\lambda_{\alpha\beta}} \sum_{\gamma \neq \alpha, \beta} a_\gamma \left(\frac{1}{\lambda_{\gamma\alpha}} + \frac{1}{\lambda_{\gamma\beta}} \right) \right]$$

$$-2\pi i R \left[(\sum_{\gamma=1}^{n} a_\gamma)(\sum_{\beta=1}^{n} s_\beta) - \frac{1}{2} \sum_{\gamma \neq \beta} \frac{a_\gamma \lambda_\gamma s_\beta}{\lambda_{\gamma\beta}} \right]. \tag{25}$$

Here α, β, γ run from 1 to n.

The above discussion can easily be generalized to the cases with N_f hypermultiplets, corresponding to N_f D sixbranes or semi-infinite D fourbranes. To illustrate this let us consider a situation where the N_f of semi-infinite D fourbranes are attached to the right hand side of the second NS fivebrane. The curve for such a configuration had already been written down[14]. In the parametric form, it is given by

$$v = \frac{(\lambda - \lambda_+)(\lambda - \lambda_-)}{\mu\lambda},$$

$$w = \lambda,$$

$$t = \mu^{-N_c} \lambda^{N_c - N_f} (\lambda - \lambda_+)^r (\lambda - \lambda_-)^{N_f - r}, \tag{26}$$

where $\lambda_\pm$ are the two solutions of $v = 0$, and μ is bare mass for the adjoint chiral multiplet. Employing the same method as above, we get following superpotential:

$$W = 2\pi i \frac{R\lambda_+ \lambda_-}{\mu} [2(N_c - N_f) + (N_f - r) + r],$$

$$= 2\pi i \frac{R\lambda_+ \lambda_-}{\mu} (2N_c - N_f). \tag{27}$$

This precisely agrees with the minimum value of $N = 1$ gauge theory superpotential

$(N_f < N_c)$ given as follows[20, 14];

$$W_{\text{eff}} = (N_c - N_f) \left(\frac{\Lambda_{N=1}^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)} + \frac{1}{2\mu} \left(\text{Tr}(M^2) - \frac{1}{N_c}(\text{Tr}M)^2 \right), \qquad (28)$$

where $\Lambda_{N=1}$ is the dynamical scale of $N = 1$ SQCD. More explicitly,

$$W_{\text{Gauge-theory}} = -\frac{1}{4\pi i} W_{\text{M-theory}}, \qquad (29)$$

with the identification $\lambda_\pm = m_\pm$, where $m_\pm$ are the only two possibly different eigenvalues of the meson matrix M whose elements are $\tilde{Q}_i^a Q_b^i$. This is the first concrete evidence that Witten's proposal for the superpotential is correct. There is an explicit rotation angle dependence through $\mu \sim \tan\theta$. The dependence on r comes only through $\lambda_\pm$.

4 Discussion

In this paper we found a parametric representation of the brane configurations corresponding to the $N = 1$ SQCD. Extending the Witten's method, we developed a general formalism to calculate the superpotentials and compared with known gauge theory results.

The comparison with gauge theory result might be interesting when we consider gauge theories for product groups. As mentioned earlier, one motivation for this comes from the study of configurations of k 'coincident' NS fivebranes linked to an 'coincident' NS fivebrane linked by N_c D fourbranes, and have the form of the superpotential for that configuration[5]. The gauge theory has $SU(N_c)$ gauge symmetry. Now when the NS fivebranes separate it corresponds to, in the field theory, resolution of the singularity in the superpotential of the form $W = X^{k+1}$+lower powers[22]. Then the gauge symmetry is spontaneously broken to $SU(N_c) \rightarrow \prod SU(r_i)$, where $\sum_i r_i = N_c$. Of course the detailed form of the potential depends on the deformation parameters away from the coincident limit.

Also more thorough understanding is necessary on the role of the complex volume form which appears in the definition of the superpotential. The implication

of the superpotential in relation to the symmetry enhancement when the branes collapse is not fully understood. These are currently being investigated.

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